EPHOU-95002
HUPS-94-5
hep-th/9503190
March 1995

Integrability of Schwinger-Dyson Equations in 2D Quantum Gravity and $c < 1$ Non-critical String Field Theory

Ryuichi NAKAYAMA *and Toshiya SUZUKI †

Department of Physics, Faculty of Science,
Hokkaido University, Sapporo 060, Japan

Abstract

We investigate the integrability of the Schwinger-Dyson equations in $c = 1 - \frac{6}{m(m+1)}$ string field theory which were proposed by Ikehara et al as the continuum limit of the Schwinger-Dyson equations of the matrix chain model. We show the continuum Schwinger-Dyson equations generate a closed algebra. This algebra contains Virasoro algebra but does not coincide with W_∞ algebra. We include in the Schwinger-Dyson equations a new process of removing from the loop boundaries the operator $\mathcal{H}(\sigma)$ which locally changes the spin configuration. We also derive the string field Hamiltonian from the continuum Schwinger-Dyson equations. Its form is universal for all $c = 1 - \frac{6}{m(m+1)}$ string theories.

*nakayama@particle.phys.hokudai.ac.jp
†tsuzuki@particle.phys.hokudai.ac.jp

String field theory [1] seems to be the most natural framework for studying the non-perturbative properties of string theory. Recently a new class of string field theory based on the transfer matrix formalism [2] for 2d quantum gravity was proposed in [3]. In this string field theory the geodesic distance from the boundaries is used as a time variable. By using this time coordinate the world sheet of c = 0 string is cut into time slices and then decomposed into vertices and propagators. This string field theory is called string field theory in the temporal gauge[4][5].

This decomposition is also possible even when matter degrees of freedom are put on the worldsheet. The simplest model of string with matter is the Ising model on a random surface, *i.e.*, c = 1/2 string. The partition function and the loop amplitudes of the Ising model on a dynamically triangulated surface [6] are defined in terms of the two-matrix model[7] and the continuum theory is obtained by the double scaling limit [8], which also enables us to discuss the summation of string perturbation series. The most effective method for non-perturbative investigation of the loop amplitudes will be the Schwinger-Dyson equations (SDEs). The SDEs in the matrix model determine the loop amplitudes completely.

The string field theory can be constructed in such a way that the continuum version of the matrix model SDEs are derived from the SDEs in string field theory. In [9] the continuum limit of the SDEs in the two-matrix model was derived under some assumptions and the string field Hamiltonian was inferred from these equations. Such assumptions were then justified by showing that W_3 constraints [10] [11] [12] can be derived from the continuum version of the matrix model SDEs. These results were also extended to c = 1 − 6/(m(m+1)) string.

The purpose of this letter is to investigate the integrability of the SDEs proposed in [9]. The SDEs of c = 0 string are so-called Virasoro constraints on the partition function [10] [11] and these SDEs are integrable because Virasoro algebra closes. The SDEs for c = 1 − 6/(m(m+1)) string proposed in [9] are more complicated than those for c = 0 string and the integrability of them is not obvious. We will show these SDEs are indeed integrable by exhibiting the algebra generated by these SDEs. This will give another justification for the assumptions in [9]. Contrary to our expectations, however, this algebra turns out to be not W_∞ but a larger one. Furthermore this algebra does not seem to contain W_∞ as a subalgebra. We will also include in the SDEs those terms corresponding to a process of removing the operator $\mathcal{H}(\sigma)$ from the boundary loops, which were not taken into account in [9]. This operator changes the configuration of spins locally.[9] We will then write down the Hamiltonian for c = 1 − 6/(m(m+1)) string in a more explicit way than in [9], where the meaning of the summation over conformal field theory (CFT) states $|v>$ in the Hamiltonian was not completely specified.

In [13] another type of c = 1/2 string field theory was constructed by changing

the definition of the time coordinate in such a way that the string does not cross the domain walls. It was shown the SDEs generate decoupled Virasoro algebras. In [14] and [15] temporal-gauge string field theory was extended to include open string fields in two different ways. It was also pointed out in [16] that the string field Hamiltonian can be derived from the stochastic quantization of the matrix model. The string field Hamiltonian was also deduced from the matrix model in [17].

Let us consider a two-matrix model defined by an action

$$S(A,B) = Ntr(\frac{1}{2}A^2 + \frac{1}{2}B^2 - cAB - \frac{1}{3}\lambda(A^3 + B^3)), \tag{1}$$

where A and B are N by N hermitian matrices and c and λ are constants. This matrix model is known to provide a formal perturbative definition of c = 1/2 non-critical string.[7] Such a string can be realized by putting the Ising model on a dynamically triangulated surface. When one assigns the vertices λA^3 and λB^3 to triangles on which up and down spins reside, respectively, the free energy of the matrix model (1) gives the partition function of the Ising model on a random surface.

The loop amplitudes in which all spins on the loop boundaries are up are expressed by correlation functions of $W_0(m) = \frac{1}{N}trA^m$. Because the action (1) mixes A and B, however, the SDEs in the two-matrix model do not close within this kind of loops but loops with mixed spin configurations appear. Therefore we have to consider correlation functions of the operator

$$W_n(m_1, m_2, \cdots, m_n) = \frac{1}{N}tr(A^{m_1}BA^{m_2}B\cdots A^{m_n}B) \qquad (n \geq 1). \tag{2}$$

This operator represents a loop on which almost all spins are up and only n non-adjacent spins are down. By considering a correlation function of $(1/N)tr(t^a A^{m_1}BA^{m_2}B\cdots BA^{m_s})$ and W_n's, where $\{t^a | a = 1, \cdots, N^2\}$ is a basis of Lie algebra $u(N)$, and using the fact that this correlation function is invariant under an infinitesimal shift $\delta A = \epsilon t^a$ of the integration variable A, we obtain an SDE for the loops (2). This SDE describes a process of taking away one triangle from a boundary and contains terms which correspond to the following processes.

1. The boundary loop splits into two.

2. The boundary loop absorbs another boundary.

3. The spin configuration on the boundary loop changes.

In [9] it was assumed that these terms survive the continuum limit and the remaining terms which change the length of the loop drop out. With this assumption we can write down the continuum SDE. The continuum limit of a loop is described by the

length l of the loop and the state of the matter on the loop, which is a state in c = 1/2 conformal field theory (CFT). The continuum limit of the simplest loop operator $W_0(m) = \frac{1}{N} tr A^m$ is specified by the length l and the state $|+>$ in c = 1/2 CFT that corresponds to the up spin. We will denote it by $\hat{w}_0(l)$. The continuum limit of (2) is specified by the lengths $\{l_1, l_2, \cdots, l_n\}$ of the segments of $|+>$ state and will be denoted by $\hat{w}_n(l_1, l_2, \cdots, l_n)$. This can be obtained from $\hat{w}_0(l_1 + \cdots l_n)$ by insertion of the local operator $\mathcal{H}(\sigma)$, which flips the spin locally, at n distinct points of the loop boundary.[9] To write down the continuum SDE we introduce the source functions $J_0(l)$ for $\hat{w}_0(l)$ and $J_n(l_1, l_2, \cdots, l_n)$ for $\hat{w}_n(l_1, l_2, \cdots, l_n)$. Then the SDEs in the continuum limit should be

$$T_n(l_1, l_2, \cdots, l_n) Z[J] \approx 0 \quad (n \geq 1), \tag{3}$$

where $Z[J] = Z[J_0, J_1, J_2, \ldots]$ is the generating functional of the disconnected loop amplitudes. T_n is a functional differential operator with respect to J's and given by

$$\begin{aligned} T_1(l) \;=\; & \int_0^l dl' D_0(l') D_0(l - l') + \int_0^\infty dl' J_0(l') l' D_0(l + l') \\ & + \sum_{m=1}^\infty \sum_{j=1}^m \int_0^\infty dl'_1 \cdots \int_0^\infty dl'_m J_m(l'_1, \cdots, l'_m) l'_j D_m(l'_1, \cdots, l'_{j-1}, l'_j + l, l'_{j+1}, \cdots, l'_m) \\ & \quad + D_1(l), \end{aligned} \tag{4}$$

$$\begin{aligned} T_n(l_1, \cdots, l_n) \;=\; & \int_0^{l_1} dl' D_0(l') D_{n-1}(l_n + l_1 - l', l_2, \cdots, l_{n-1}) \\ & + \sum_{k=2}^{n-1} \int_0^{l_k} dl' D_{k-1}(l_1 + l', l_2, \cdots, l_{k-1}) D_{n-k}(l_n + l_k - l', l_{k+1}, \cdots, l_{n-1}) \\ & + \int_0^{l_n} dl' D_{n-1}(l_1 + l', l_2, \cdots, l_{n-1}) D_0(l_n - l') \\ & + \int_0^\infty dl' J_0(l') l' D_{n-1}(l_n + l_1 + l', l_2, \cdots, l_{n-1}) \\ & + \sum_{m=1}^\infty \sum_{j=1}^m \int_0^\infty dl'_1 \cdots \int_0^\infty dl'_m J_m(l'_1, \cdots, l'_m) \int_0^{l'_j} dl'' \\ & \quad \cdot D_{m+n-1}(l'_1, \cdots, l'_{j-1}, l_1 + l'', l_2, \cdots, l_{n-1}, l_n + l'_j - l'', l'_{j+1}, \cdots, l'_m) \\ & \qquad + D_n(l_1, \cdots, l_n), \qquad (n \geq 2). \end{aligned} \tag{5}$$

$D_0(l)$ and $D_n(l_1, \cdots, l_n)$ are the functional derivatives defined by $D_0(l) J_0(l') = \delta(l - l')$ and $D_n(1_1, \cdots, l_n) J_m(l'_1, \cdots, l'_m) = (1/n)\delta_{nm}(\delta(1_1 - l'_1) \cdots \delta(l_n - l'_n) +$ cyclic permutations). Here the string coupling constant g is suppressed for simplicity. These SDEs describe the above three processes.(figs 1 and 2) The symbol ≈ 0 means that the left hand side is equal to a sum of terms proportional to the products of the delta functions $\delta(l)$ and $\delta(l_j)$ and their derivatives. These terms represent the following two processes.

4. When a triangle is taken away at the point where the spin is down, the spin flips up.

5. A loop with vanishing length disappears.

From the point of view of string field theory process 5 is expressed by the tadpole terms.[9] Process 4 is a new one that was not taken into account in [9]. Later we will incorporate such a process into (5).

In [9] it was shown that a combination of a subset of SDE(3)

$$\begin{aligned} T_1(l)Z[J]|_{(J_n=0,n\geq 1)} &\approx 0, \\ T_2(l,0)Z[J]|_{(J_n=0,n\geq 1)} &\approx 0 \end{aligned} \tag{6}$$

and the 'null' condition

$$D_2(l,0)Z[J]|_{(J_n=0,n\geq 1)} \approx 0 \tag{7}$$

is equivalent to the W_3 constraints [10] [11] [12]. We can show that the commutator of $T_n(l_1,\cdots,l_n)$ and $D_m(l'_1,\cdots,l'_{m-1},0)$ is given by a linear combination of terms of the form $D_{n+m-1}(l''_1,\cdots,l''_{n+m-2},0)$. Hence the 'null' condition $D_m(l_1,\cdots,l_{m-1},0)Z \approx 0$ and the SDEs are compatible.

A central issue of this letter is the integrability of (3). SDEs in c = 0 string can be succinctly written as Virasoro constraints [10] [11] and the closure of Virasoro algebra ensures the integrability of SDEs. For c = 1/2 string this algebra is replaced by W_3 algebra [10] [11] [12] and we might expect (4) and (5) generate W_3 algebra. In [13] an alternative definition of the time coordinate is chosen in c = 1/2 string field theory and the resulting algebra is shown to be decoupled Virasoro algebras. We study whether T_n generates a closed algebra, and if it does, what the algebra is. Here processes 4 and 5 will be ignored and later we will take only process 4 into account. The calculation is straightforward and we will present only the results of long calculation.

$$[T_1(l),T_1(\tilde{l})] = (l-\tilde{l})T_1(l+\tilde{l}), \tag{8}$$

$$\begin{aligned} [\ T_n(l_1,\cdots,l_n),T_1(l)\ ] &= \sum_{j=1}^{n} l_j T_n(l_1,\cdots,l_{j-1},l_j+l,l_{j+1},\cdots,l_n) \\ &\quad -\int_0^l dl' T_n(l_1+l',l_2,\cdots,l_{n-1},l_n+l-l') \\ &\qquad (n\geq 2), \end{aligned} \tag{9}$$

$$\begin{aligned} &[\ T_n(l_1,\cdots,l_n),T_m(\tilde{l}_1,\cdots,\tilde{l}_m)\ ] \\ = &\int_0^{l_1} dl' T_{n+m-1}(\tilde{l}_1+l',\tilde{l}_2,\cdots,\tilde{l}_{m-1},\tilde{l}_m+l_1-l',l_2,\cdots,l_n) \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{j=2}^{n-1}\int_0^{l_j} dl' T_{n+m-1}(l_1,\cdots,l_{j-1},\tilde{l}_1+l',\tilde{l}_2,\cdots,\tilde{l}_{m-1},\tilde{l}_m+l_j-l',l_{j+1},\cdots,l_n) \\
&+ \int_0^{l_n} dl' T_{n+m-1}(l_1,\cdots,l_{n-1},\tilde{l}_1+l',\tilde{l}_2,\cdots,\tilde{l}_{m-1},\tilde{l}_m+l_n-l') \\
&\quad -(n\leftrightarrow m, l_j\leftrightarrow \tilde{l}_j) \qquad (n,m\geq 2). \qquad (10)
\end{aligned}
$$

Hence the algebra closes and SDEs (3) are integrable. The algebra by itself is quite intriguing. While (8) is Virasoro algebra, the whole algebra including the other two, (9) and (10), is a new one and there seems to be no simple relationship to W_3 or W_∞ algebra.

Let us next incorporate process 4. This will be done by adding to T_n some terms proportional to D_{n-2} so that the algebra (8) $\sim$ (10) remains unchanged. We replace $T_n(n\geq 2)$ by the following operator $\tilde{T}_n$.

$$
\tilde{T}_2(l_1,l_2) = T_2(l_1,l_2) + \{a+b(\partial_{l_1}+\partial_{l_2})\}\{(\delta(l_1)+\delta(l_2))D_0(l_1+l_2)\}, \qquad (11)
$$

$$
\begin{aligned}
\tilde{T}_n(l_1,\cdots,l_n) = T_n(l_1,\cdots,l_n) + &\\
\{a+b(\partial_{l_1}+\partial_{l_n})\}\{ &\ \delta(l_1)\ D_{n-2}(l_n+l_1+l_2,l_3,\cdots,l_{n-1}) \\
+ &\ \delta(l_n)\ D_{n-2}(l_{n-1}+l_n+l_1,l_2,\cdots,l_{n-2})\} \\
&\qquad (n\geq 3). \qquad (12)
\end{aligned}
$$

T_1 is left unchanged: $\tilde{T}_1(l)=T_1(l)$. The second terms in (11) and (12) describe process 4.(figs 3 and 4) Here a and b are constants. These constants are proportional to positive powers of the cosmological constant t and the powers are determined as follows. The scaling dimension of the disk amplitude $<\hat{w}_0(l)>_0$ can be estimated by KPZ-DDK argument [18] [19] to be $L^{-7/3}$, where L stands for the dimension of the boundary length. $<\hat{w}_n(l_1,\cdots,l_n)>_0$ can be obtained from $<\hat{w}_0(l)>_0$ by insertion of the operator $\mathcal{H}(\sigma)$, which has dimension $L^{-4/3}$, at n distinct points of the boundary.[9] Hence $D_n(l_1,\cdots,l_n)$ and $T_n(l_1,\cdots,l_n)$ have dimension $L^{-(7+4n)/3}$. Because t has dimension L^{-2}, we find a and b are proportional to $t^{5/6}$ and $t^{1/3}$, respectively. Higher derivative terms are not included because the coefficients will become negative powers of t. It is non-trivial but straightforward to show $\tilde{T}_n(l_1,\cdots,l_n)$ also satisfies (8) $\sim$ (10).

Process 5 corresponds to the tadpole terms in the view point of string field theory.[9] Such terms contain the procuct of delta functions $\delta(l_j)$ and their derivatives multiplied by a functional of the source functions J_n. [20] It is rather difficult to determine those terms solely by integrability conditions and we will not pursue this problem here.

We now turn to string field theory. The continuum version of the matrix model SDE is closely related to the string field Hamiltonian and general form of the Hamiltonian in $c<1$ string field theory was presented in [9]. The expression of the Hamiltonian

was, however, rather formal because the meaning of the summation over CFT states $|v>$ was not completely specified. Here we will write down the string field Hamiltonian more explicitly. For detailed discussion of the string field theory in the temporal gauge we refer the reader to [9] [3] [13] [21].

General prescription is to express the generating functional in the form

$$\begin{aligned} Z[J] &= \lim_{D\to\infty} <0|e^{-DH}e^{S_{source}}|0>, &(13)\\ S_{source} &= \int_0^\infty dl J_0(l)\Psi_0^\dagger(l) + \sum_{n=1}^\infty \int_0^\infty dl_1\cdots\int_0^\infty dl_n J_n(l_1,\cdots,l_n)\Psi_n^\dagger(l_1,\cdots,l_n). &(14)\end{aligned}$$

Here H is the string field Hamiltonian and D is the geodesic distance from the boundaries. $\Psi_0^\dagger(l)$ and $\Psi_n^\dagger(l_1,\cdots,l_n)$ are the creation operators of the loops $\hat{w}_0(l)$ and $\hat{w}_n(l_1,\cdots,l_n)$, respectively, and these satisfy the usual commutation relations with the corresponding annihilation operators Ψ_0 and Ψ_n.

$$\begin{aligned} [\ \Psi_0(l),\Psi_0^\dagger(l')\ ] &= \delta(l-l'),\\ [\ \Psi_n(l_1,\cdots,l_n),\Psi_n^\dagger(l_1',\cdots,l_n')\ ] &= \frac{1}{n}\{\delta(l_1-l_1')\cdots\delta(l_n-l_n')\\ &\qquad +\text{cyclic permutations}\}. \qquad (15)\end{aligned}$$

The vacuum state $|0>$ is defined by

$$\Psi_0(l)|0>=\Psi_n(l_1,\cdots,l_n)|0>=0. \qquad (16)$$

Then the string field SDE can be obtained as the condition for the existence of the limit (13) and reads

$$-\lim_{D\to\infty}\frac{\partial}{\partial D}<0|e^{-DH}e^{S_{source}}|0>=\lim_{D\to\infty}<0|e^{-DH}He^{S_{source}}|0>=0. \qquad (17)$$

Now Ψ_n and $\Psi_n^\dagger$ in H can be replaced by J_n and D_n, respectively, and (17) can be rewritten as a differential equation for $Z[J]$.

$$\hat{H}Z[J]=0 \qquad (18)$$

Here $\hat{H}$ is a differential operator with respect to J's. The string field SDE (18) and the matrix model SDE will be equivalent under suitable boundary conditions [3] if $\hat{H}$ is chosen as follows.

$$\begin{aligned} \hat{H} &= \int_0^\infty dl J_0(l) l\tilde{T}_1(l)\\ &\quad +\sum_{n=1}^\infty n\int_0^\infty dl_1\cdots\int_0^\infty dl_{n+1}J_n(l_1+l_{n+1},l_2,\cdots,l_n)\tilde{T}_{n+1}(l_1,\cdots,l_{n+1}). \qquad (19)\end{aligned}$$

This yields the following string field Hamiltonian.

$$H = H_1 + H_2 + H_3 + H_4 + H_5, \tag{20}$$

$$\begin{aligned}
H_1 &= \int_0^\infty dl_1 \int_0^\infty dl_2 \Psi_0^\dagger(l_1)\Psi_0^\dagger(l_2)\Psi_0(l_1+l_2)(l_1+l_2) \\
&+2\sum_{n=1}^\infty n \int_0^\infty dl \int_0^\infty dl_1 \cdots \int_0^\infty dl_n \Psi_0^\dagger(l)\Psi_n^\dagger(l_1,\cdots,l_n)\Psi_n(l_1+l,l_2,\cdots,l_n)l_1 \\
&+\sum_{n=1}^\infty\sum_{m=1}^\infty (n+m)\int_0^\infty dl \int_0^\infty dl' \int_0^\infty dl_1 \cdots \int_0^\infty dl_n \int_0^\infty dl_1' \cdots \int_0^\infty dl_m' \\
&\Psi_n^\dagger(l_1+l,l_2,\cdots,l_n)\Psi_m^\dagger(l_1'+l',l_2',\cdots,l_m')\Psi_{n+m}(l_1+l',l_2,\cdots,l_n,l_1'+l,l_2'\cdots,l_m'),
\end{aligned}$$

$$\begin{aligned}
H_2 &= g\int_0^\infty dl_1 \int_0^\infty dl_2 \Psi_0^\dagger(l_1+l_2)\Psi_0(l_1)\Psi_0(l_2)l_1l_2 \\
&+2g\sum_{n=1}^\infty n \int_0^\infty dl \int_0^\infty dl_1 \cdots \int_0^\infty dl_n \Psi_n^\dagger(l_1+l,l_2,\cdots,l_n)\Psi_n(l_1,\cdots,l_n)\Psi_0(l)ll_1 \\
&+g\sum_{n=1}^\infty\sum_{m=1}^\infty nm \int_0^\infty dl \int_0^\infty dl' \int_0^\infty dl_1 \cdots \int_0^\infty dl_n \int_0^\infty dl_1' \cdots \int_0^\infty dl_m' \\
&\Psi_{n+m}^\dagger(l_1+l,l_2,\cdots,l_n,l_1'+l',l_2',\cdots,l_m')\Psi_n(l_1+l',l_2,\cdots,l_n)\Psi_m(l_1'+l,l_2',\cdots,l_m'),
\end{aligned}$$

$$\begin{aligned}
H_3 &= \int_0^\infty dl \Psi_1^\dagger(l)\Psi_0(l)l \\
&+\sum_{n=1}^\infty n \int_0^\infty dl_1 \cdots \int_0^\infty dl_{n+1}\Psi_{n+1}^\dagger(l_1,\cdots,l_{n+1})\Psi_n(l_1+l_{n+1},l_2,\cdots,l_n),
\end{aligned}$$

$$\begin{aligned}
H_4 &= 2a\ [\ \int_0^\infty dl \Psi_0^\dagger(l)\Psi_1(l) \\
&+\sum_{n=2}^\infty n \int_0^\infty dl_1 \cdots \int_0^\infty dl_n \Psi_{n-1}^\dagger(l_n+l_1,l_2,\cdots,l_{n-1})\Psi_n(l_1,\cdots,l_n)\ ] \\
&+b\ [\ 2\int_0^\infty dl \{\partial_l\Psi_0^\dagger(l)\cdot\Psi_1(l) - \Psi_0^\dagger(l)\cdot\partial_l\Psi_1(l)\} \\
&+\sum_{n=2}^\infty n \int_0^\infty dl_1 \cdots \int_0^\infty dl_n \{(\partial_{l_1}+\partial_{l_n})\Psi_{n-1}^\dagger(l_n+l_1,l_2,\cdots,l_{n-1})\cdot\Psi_n(l_1,\cdots,l_n) \\
&-\Psi_{n-1}^\dagger(l_n+l_1,l_2,\cdots,l_{n-1})\cdot(\partial_{l_1}+\partial_{l_n})\Psi_n(l_1,\cdots,l_n)\}\ ]\ ,
\end{aligned}$$

$$\begin{aligned}
H_5 &= \int_0^\infty dl \rho_0(l)\Psi_0(l) \\
&+\sum_{n=1}^\infty \int_0^\infty dl_1 \cdots \int_0^\infty dl_n \rho_n(l_1,\cdots,l_n)\Psi_n(l_1,\cdots,l_n).
\end{aligned} \tag{21}$$

Here we introduced the string coupling constant g, which has dimension $L^{-14/3}$. The dimensions of fields are $[\Psi_n^\dagger] = L^{-(7+4n)/3} \quad (n \geq 0)$, $[\Psi_0] = L^{4/3}$ and $[\Psi_n] = L^{(7+n)/3} \quad (n \geq 1)$. The tadpole terms H_5 are also included in H although the explicit form of ρ_n is left undetermined. It is clear H_n describes process n.

The string states $\Psi_0^\dagger(l)|0>$ and $\Psi_n^\dagger(l_1, \cdots, l_n)|0>$ which appear in the above string field theory do not cover the whole space of string states. There exist more general mixed spin configurations

$$\frac{1}{N} tr(A^{m_1} B^{n_1} A^{m_2} B^{n_2} \cdots A^{m_s} B^{n_s}) \qquad (m_i, n_i \geq 1). \tag{22}$$

Nonetheless SDEs (3) satisfy the integrability condition and the Hamiltonian (20) defines a consistent string field theory. It should also be possible to write down the continuum SDE for the loops (22) by a straightforward extension of the present work and to obtain the string field theory for the whole string space. In a sense we successfully truncated the string space. As a result of this truncation the world sheet in our string field theory does not contain two-dimensional domains of down spins. Down spins are confined into one-dimensional regions, trajectories of the locations of the inserted operator $\mathcal{H}$. This is in sharp contrast to the world sheet in another formulation of the c= 1/2 string field theory in [13].

It may be important to stress the necessity of the new process 4. The string field Hamiltonian describes the time evolution of loops on the world sheet. Suppose sweeping the world sheet by loops of constant time. When the loop of constant time meets a one-dimensional domain of down spins along the way, the operator $\mathcal{H}$ is inserted (process 3). This operator will be present for a while, but eventually it will disappear at the other end of the one-dimensional domain. This is the process 4. Hence it is quite plausible to incorporate such a process into the Hamiltonian. We have shown this is indeed possible to do without changing the algebra (8) $\sim$ (10).

A remarkable fact is that the Hamiltonian (20) is universal for all c = 1 − 6/(m(m+1)) strings. c = 1 − 6/(m(m+1)) string can be realized by the (m − 1)-matrix chain model. The matrices M_i are labeled by an integer $i(= 1, \cdots, m-1)$ and the action has the form

$$S(M_1, \cdots, M_{m-1}) = \frac{1}{N} tr(\sum_{i=1}^{m-1} V_i(M_i) - c \sum_{i=1}^{m-2} M_i M_{i+1}). \tag{23}$$

It is obvious that if we consider correlation functions of the operator

$$\frac{1}{N} tr(M_1^{m_1} M_2 M_1^{m_2} M_2 \cdots M_1^{m_n} M_2), \tag{24}$$

we obtain the same SDEs as (3). Hence the same string field Hamiltonian (20). The constants a and b are now given by

$$a = a_0 t^{1/m+1/2} \text{ and } b = b_0 t^{1/m}, \tag{25}$$

where a_0 and b_0 do not depend on the cosmological constant t. Because the string field Hamiltonian is the same for any c, it might be possible to go beyond the c=1 barrier by investigations along this line.

The constants a_0 and b_0 in (25) remain unknown. The tadpole terms ρ_n are undetermined, either. To determine these we need to compute various amplitudes of $\hat{w}_n$. Necessity of relative angle integrations in matrix chain models hinder the calculation of loop amplitudes with mixed spin configurations on the loop boundaries and these problems are left to the future investigations.

To recapitulate, we demonstrated that continuum SDEs (3) are integrable and they generate a closed algebra (8)$\sim$(10). Apparently this algebra does not seem to be related to W_∞ but it will certainly be important to pursue its connection to W_∞ further. We also derived string field Hamiltonian from SDEs(3) and this Hamiltonian was found to be universal for all $c = 1 - 6/(m(m+1))$ strings. Because there are already two different choices of the time coordinate[13][9] in constructing $c < 1$ string field theory, there may exist another choice in terms of which SDEs and string field Hamiltonian have a manifest W_∞ structure.

R. N. thanks N. Ishibashi and H. Kawai for discussions.

References

[1] M.Kaku and K.Kikkawa, Phys. Rev. **D10** (1974) 1110, 1823;
W.Siegel, Phys. Lett. **B151** (1985) 391, 396;
E.Witten, Nucl. Phys. **B268** (1986) 253;
Hata, K.Itoh, T.Kugo, H.Kunitomo and K.Ogawa, Phys. Lett. **B172** (1986) 186, 195;
A.Neveu and P.West, Phys. Lett. **B168** (1986) 192;
M.Kaku, Int. J. Mod. Phys. **A9** (1994) 139.

[2] H. Kawai, N. Kawamoto, T. Mogami and Y. Watabiki, Phys. Lett. **306B** (1993) 19.

[3] N. Ishibashi and H. Kawai, Phys. Lett. **B314** (1993) 190.

[4] M.Fukuma, N.Ishibashi, H.Kawai and M.Ninomiya, Nucl. Phys. **B427** (1994) 139.

[5] R.Nakayama, Phys. Lett. **B325** (1994) 347.

[6] V.A. Kazakov, I.K. Kostov and A.A. Migdal, Phys. Lett. **66** (1991) 2051;
J. Ambjørn, B. Durhuus and J. Fröhlich, Nucl. Phys. **B257** [FS14] (1985) 433;
F. David, Nucl. Phys. **B257** [FS14] (1985) 543.

[7] V.A.Kazakov, Phys. Lett. **A119** (1986) 140;
D.V.Boulatov and V.A.Kazakov, Phys. Lett. **B186** (1987) 379.

[8] E. Brézin and V. Kazakov, Phys. Lett. **B236** (1990) 144;
M. Douglas and S. Shenker, Nucl. Phys. **B335** (1990) 635;
D. J. Gross and A. Migdal, Phys. Rev. Lett. **64** (1990) 127.

[9] M. Ikehara, N. Ishibashi, H. Kawai, T. Mogami, R. Nakayama and N. Sasakura, Phys. Rev. **D50** (1994) 7467.

[10] M. Fukuma, H. Kawai and R. Nakayama, Int. J. Mod. Phys. **A6** (1991) 1385.

[11] R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. **B348** (1991) 435.

[12] E.Gava and K.S.Narain, Phys. Lett. **B263** (1991) 213.

[13] N. Ishibashi and H. Kawai, Phys. Lett. **B322** (1994) 67.

[14] T. Mogami, *Temporal-gauge String Field with Open Strings*, preprint KEK-TH-426 (Dec 1994).

[15] I.K. Kostov, *Loop Space Hamiltonian for $c \leq 1$ Open Strings*, Saclay preprint SPht/95-001.

[16] A.Jevicki and J.Rodrigues, Nucl. Phys. **B421** (1994) 278.

[17] Y.Watabiki, *Construction of Non-critical String Field Theory by Transfer Matrix Formalism in Dynamical Triangulation*, preprint, INS-1017, (Dec 1993).

[18] V.G. Knizhnik, A.M. Polyakov and A.B. Zamolodchikov, Mod. Phys. Lett. **A3** (1988) 819.

[19] J. Distler and H. Kawai, Nucl. Phys. **B321** (1989) 509;
F. David, Mod. Phys. Lett. **A3** (1988) 1651.

[20] M. Ikehara, *The Continuum Limit of the Schwinger-Dyson Equations of the One and Two Matrix Model with Finite Loop Length*, preprint KEK-TH-424 (Dec 1994).

[21] M. Ikehara, N. Ishibashi, H. Kawai, T. Mogami, R. Nakayama and N. Sasakura, *A Note on String Field Theory in the Temporal Gauge*, proceedings of tne workshop *Quantum Field Thoery, Integrable Models and Beyond*, Yukawa Institute for Theoretical Physics, Kyoto University, 14-18 Feb 1994, preprint KEK-TH-411 (Aug 1994).

Figure Captions

Fig.1 Processes $1 \sim 3$ involved in SDE $T_1(l)Z[J] \approx 0$. The cross on the left hand side represents the position of deformation. The solid curve stands for the portion of the loop on which the spins are up and the dot for insertion of $\mathcal{H}$ which changes the configuration of spins.

Fig.2 Processes $1 \sim 3$ involved in SDE $T_n(l_1, \cdots, l_n)Z[J] \approx 0$ $(n \geq 2)$.

Fig.3 Process 4 to be added to $T_2(l_1, l_2)Z[J]$. If the position of deformation coincides with the position of $\mathcal{H}$, *i.e.*, $l_1 = 0$, $\mathcal{H}$ may be removed.

Fig.4 Process 4 to be added to $T_n(l_1, \cdots, l_n)Z[J]$ $(n \geq 3)$.

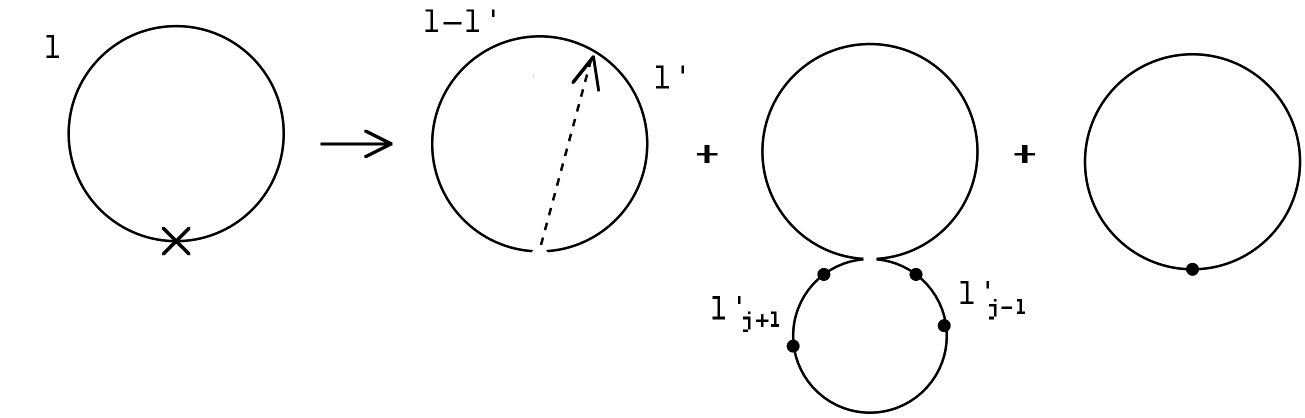

Fig.1

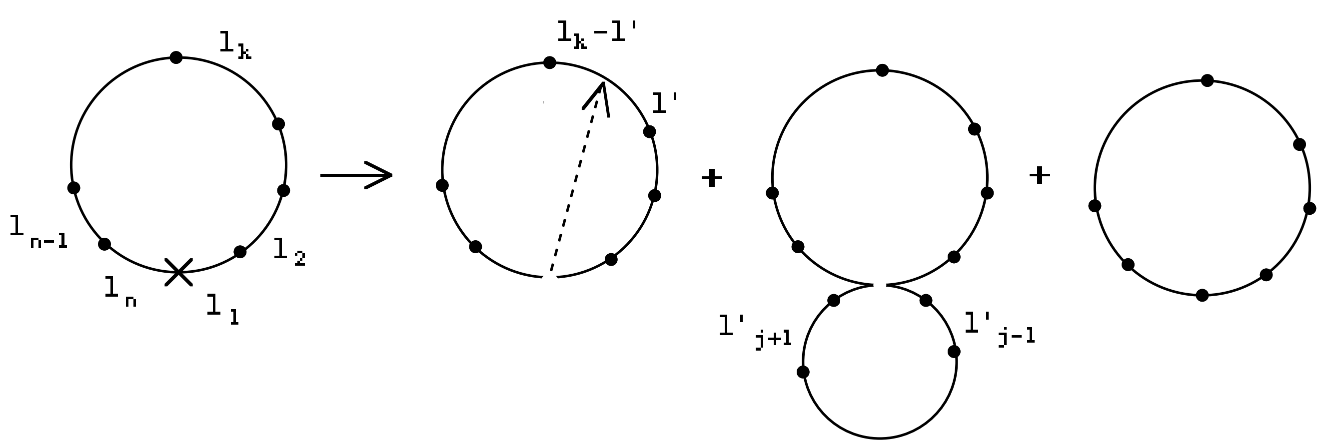

Fig.2

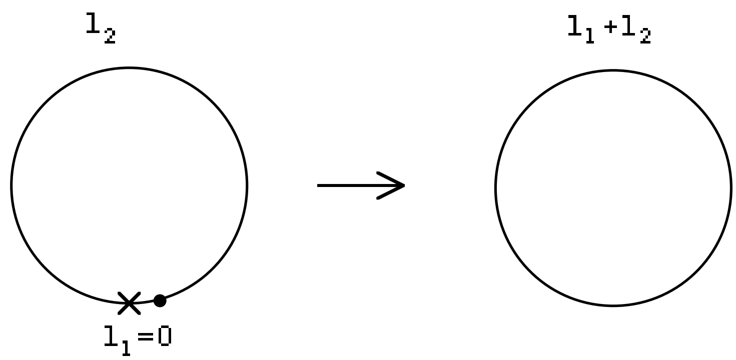

Fig.3

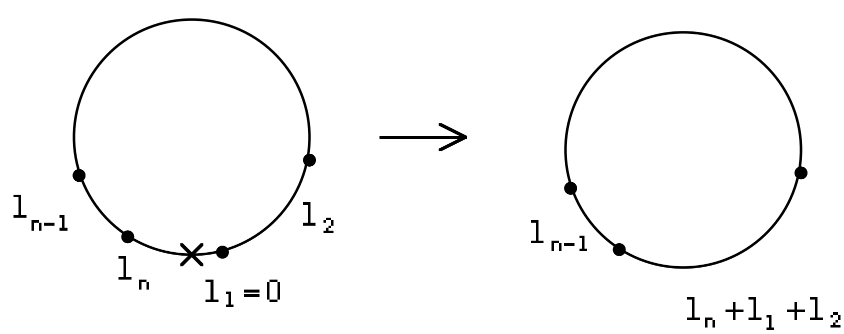

Fig.4